OSCILLATORY INTEGRALS
OPEN PROBLEMS IN NUMBER THEORY
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ZEÉV RUDNICK

Contents

1. Oscillatory integrals

The Fourier transform $\widehat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} dx$ is an example of an oscillatory integral. The general form that we take is

$$I(\lambda) := \int_{-\infty}^{\infty} A(x) e^{i\lambda\phi(x)} dx$$

where the phase function $\phi(x)$ is assumed to be real, and the amplitude $A(x)$ is assumed to be compactly supported, or at least in the Schwartz space $\mathcal{S}$.

We will need to analyze a number of such oscillatory integrals and in particular understand their decay at infinity. The trivial bound is $O(\int |A(x)| dx)$, and we want to have some cancellation as $\lambda \to \infty$.

1.1. The principle of NON-stationary phase.

1.1. The principle of NON-stationary phase. Suppose first that A is smooth, and either

1) $|\phi'| \geq 1$

or

2) if A is compactly supported we may simply assume that $\phi' \neq 0$ has no critical points on the support of A.

Then as $\lambda \to +\infty$,

$$I(\lambda) \ll_N \frac{1}{\lambda^N}, \quad \forall N \geq 1$$

(the implied constants depend on A and ϕ).

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Proof. The proof is by repeated integration by parts. Lets take the case that A is compactly supported. Define differential operators acting on functions supported in $\operatorname{supp}(A)$:

$$L = \frac{1}{i\phi'(x)}\frac{d}{dx}, \qquad L^T = -\frac{d}{dx}\circ\frac{1}{i\phi'(x)}$$

(this makes sense since $\phi' \neq 0$ on the support of A). Integration by parts shows that for any $f, g \in \mathcal{S}$,

$$\int_{-\infty}^{\infty}(Lf)(x)g(x)dx = \int_{-\infty}^{\infty} f(x)(L^T g)(x)dx.$$

Moreover a computation shows that $L(e^{i\lambda\phi(x)}) = \lambda e^{i\lambda\phi(x)}$, so that

$$L^N(e^{i\lambda\phi(x)}) = \lambda^N e^{i\lambda\phi(x)}, \qquad \forall N \geq 1.$$

Now we have

$$I(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda\phi(x)}A(x)dx = \int_{-\infty}^{\infty}\frac{1}{\lambda^N}L^N(e^{i\lambda\phi(x)})A(x)dx = \frac{1}{\lambda^N}\int_{-\infty}^{\infty} e^{i\lambda\phi(x)}(L^T)^N(A)(x)dx$$

and taking absolute values gives

$$|I(\lambda)| \leq \frac{1}{\lambda^N}\int_{-\infty}^{\infty}|(L^T)^N(A)(x)|dx = \frac{C_N}{\lambda^N}$$

with $C_N = \int_{-\infty}^{\infty}|(L^T)^N(A)(x)|dx < \infty$ since $(L^T)^N(A)$ is also smooth and compactly supported. □

1.2. **Van der Corput's Lemma.** In the above, it was crucial to have A smooth in order to integrate by parts. When this does not hold, the result may fail. We will need the case $A = \mathbf{1}_{[a,b]}$, when

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}dx$$

We first assume that ϕ has no stationary (critical) points on $[a, b]$:

Proposition 1.1 (Van der Corput's Lemma 1)**.** *Assume that ϕ is smooth, that $\phi' \neq 0$ on $[a, b]$, and that ϕ' is monotonic. Then*

$$\Big|\int_a^b e^{i\lambda\phi(x)}dx\Big| \leq \frac{4}{\min_{x\in[a,b]}|\phi'(x)|}\cdot\frac{1}{|\lambda|}.$$

Proof. Since ϕ' is continuous and nonzero on $[a, b]$, we have

$$c := \min_{x\in[a,b]}|\phi'(x)| > 0.$$

Integrating by parts, we have

$$\begin{aligned} I(\lambda) = \int_a^b e^{i\lambda\phi(x)}dx &= \int_a^b e^{i\lambda\phi(x)} i\lambda\phi'(x)\cdot\frac{1}{i\lambda\phi'(x)}dx \\ &= e^{i\lambda\phi(x)}\frac{1}{i\lambda\phi'(x))}\Big|_a^b - \int_a^b e^{i\lambda\phi(x)}\frac{d}{dx}\Big(\frac{1}{i\lambda\phi'(x)}\Big)dx \end{aligned}$$

and therefore (WLOG take $\lambda > 0$)

$$|I(\lambda)| \leq \frac{1}{\lambda}\Big|\frac{e^{i\lambda\phi(b)}}{\phi'(b)} - \frac{e^{i\lambda\phi(a)}}{\phi'(a)}\Big| + \frac{1}{\lambda}\int_a^b\Big|\frac{d}{dx}\Big\{\frac{1}{\phi'(x)}\Big\}\Big|dx.$$

Since $|\phi'| \geq c$ on $[a,b]$, we have

$$\Big|\frac{e^{i\lambda\phi(b)}}{\phi'(b)} - \frac{e^{i\lambda\phi(a)}}{\phi'(a)}\Big| \leq \frac{1}{|\phi'(b)|} + \frac{1}{|\phi'(a)|} \leq \frac{2}{c}.$$

For the integral, since ϕ' is monotonic (and nonzero), we have $1/\phi'$ monotonic, so that $\frac{d}{dx}\left\{\frac{1}{\phi'(x)}\right\}$ has a fixed sign. Therefore

$$\int_a^b \Big|\frac{d}{dx}\Big\{\frac{1}{\phi'(x)}\Big\}\Big|dx = \Big|\int_a^b \frac{d}{dx}\Big\{\frac{1}{\phi'(x)}\Big\}dx\Big| = \Big|\frac{1}{\phi'(b)} - \frac{1}{\phi'(a)}\Big| \leq \frac{2}{c}$$

as before. Thus we find

$$|I(\lambda)| \leq \frac{4/c}{\lambda}.$$

□

We now turn to study the case when the phase function ϕ has critical (stationary) points. Assume that all critical points are non-degenerate ($\phi''(x) \neq 0$ if $\phi'(x) = 0$). By subdividing the interval[1], we can assume that either case 1 holds (no critical points) or $\phi'' \neq 0$ on the entire interval $[a,b]$. In that case we have

Proposition 1.2 (Van der Corput's Lemma 2)**.** *Let ϕ be real valued and smooth on $[a,b]$, with $\phi'' \neq 0$ on $[a,b]$. Then*

$$\Big|\int_a^b e^{i\lambda\phi(x)}dx\Big| \leq \frac{8}{\sqrt{\min_{x\in[a,b]}|\phi''(x)|}} \cdot \frac{1}{\sqrt{|\lambda|}}.$$

Proof. By replacing ϕ by $\phi/\min_{x\in[a,b]}|\phi''(x)|$ and λ by $\min_{x\in[a,b]}|\phi''(x)|\cdot\lambda$, we may assume that $|\phi''(x)| \geq 1$, and WLOG take $\phi'' \geq 1$.

Let $c \in [a,b]$ be a point where the first derivative $|\phi'|$ is minimal: $|\phi'(c)| \leq |\phi'(x)|$ for all $x \in [a,b]$. Since the second derivative is non-vanishing, it cannot be the case that c is an interior local minimum/maximum of ϕ', and hence either $\phi'(c) = 0$ or c is one of the endpoints a, b.

Assume first that $\phi'(c) = 0$, as in Figure 1. Then outside the interval $(c-\delta, c+\delta)$,

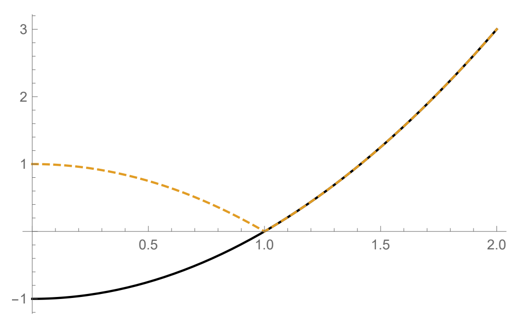

FIGURE 1. A local minimum of $|\phi'|$ where $\phi'(c) = 0$.

[1]We assume that ϕ has only finitely many critical points in $[a,b]$, which would be the case if it was real analytic.

we have $|\phi'(x)| \geq \delta$, because we assume $\phi''(x) \geq 1$ for all $x \in [a,b]$. Indeed, if $c+\delta \leq x \leq b$ then

$$\phi'(x) = \phi'(x) - \phi'(c) = \int_c^x \phi''(t)dt \geq \int_c^x 1dt = x - c \geq \delta$$

with a similar argument if $a \leq x \leq c-\delta$. Now divide the range of integration $[a,b]$ into three subintervals

$$\int_a^b e^{i\lambda\phi(x)}dx = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^b$$

On the interval $[a, c-\delta]$, use $|\phi'| \geq \delta$ and the fact that ϕ' is monotonic increasing (since $\phi'' \geq 1 > 0$) to invoke Proposition 1.1 giving

$$\Big|\int_a^{c-\delta} e^{i\lambda\phi(x)}dx\Big| \leq \frac{4}{|\min(|\phi'(x)| : x \in [a, c-\delta])} \cdot \frac{1}{\lambda} \leq \frac{4}{\delta} \cdot \frac{1}{\lambda}$$

with the same bound holding for $\int_{c+\delta}^b$.

On the middle interval $(c-\delta, c+\delta)$, just estimate trivially $|e^{i\lambda\phi(x)}| \leq 1$ giving

$$\Big|\int_{c-\delta}^{c+\delta} e^{i\lambda\phi(x)}dx\Big| \leq 2\delta$$

Thus we find

$$\Big|\int_a^b e^{i\lambda\phi(x)}dx\Big| \leq \frac{8}{\delta\lambda} + 2\delta.$$

Taking $\delta = 2/\sqrt{\lambda}$ gives

$$\Big|\int_a^b e^{i\lambda\phi(x)}dx\Big| \leq \frac{8}{\sqrt{\lambda}}.$$

It remains to deal with the case that c is one of the endpoints, say $c = a$ and $\phi'(a) \neq 0$, say $\phi'(x) \geq \phi'(a) > 0$ for all $x \in [a,b]$. Then as before $\phi'(x) \geq \delta$ for $x \in [a+\delta, b]$ since $\phi'' \geq 1$, and then the previous argument shows that

$$\Big|\int_a^b e^{i\lambda\phi(x)}dx\Big| \leq \Big|\int_a^{a+\delta} 1dx\Big| + \Big|\int_{a+\delta}^b e^{i\lambda\phi(x)}dx\Big| \leq \delta + \frac{4}{\delta\lambda} \leq \frac{4}{\sqrt{\lambda}}$$

on taking $\delta = 2/\sqrt{\lambda}$. □

Remark: A similar result holds for the case of degenerate critical points, if we assume that for some $k \geq 2$, we have $\phi^{(k)} \neq 0$ on $[a,b]$:

Proposition 1.3. *There is an absolute constant $c_k > 0$ so that for all smooth, real valued ϕ with $\phi^{(k)} \neq 0$ on $[a,b]$,*

$$\Big|\int_a^b e^{i\lambda\phi(x)}dx\Big| \leq \frac{c_k}{(\min_{x\in[a,b]} |\phi^{(k)}(x)|)^{1/k}} \cdot \frac{1}{|\lambda|^{1/k}}.$$

Exercise 1. *The Bessel function $J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{z}{2}\right)^{2m}$ admits an integral representation*

$$J_0(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iz\sin t}dt$$

Show that as $z \to +\infty$, $J_0(z) \ll 1/\sqrt{z}$.

We now include an amplitude

Corollary 1.4. *Assume that $A(x) \in C^1[a,b]$ is differentiable, that ϕ is smooth and real valued, and that $\phi^{(k)} \neq 0$ on $[a,b]$ (and if $k=1$ also that ϕ' is monotonic). Then*

$$\Big|\int_a^b A(x)e^{i\lambda\phi(x)}dx\Big| \leq \frac{c_k}{(\min_{x\in[a,b]}|\phi^{(k)}(x)|)^{1/k}} \cdot \Big(|A(b)| + \int_a^b |A'(t)|dt\Big)\frac{1}{|\lambda|^{1/k}}.$$

Proof. For notational simplicity we treat the case $k=2$. Denote

$$J_\lambda(t) = \int_a^t e^{i\lambda\phi(x)}dx$$

So that $J_\lambda(t) = e^{i\lambda\phi(t)}$. Integrating by parts, we have

$$\int_a^b A(x)e^{i\lambda\phi(x)}dx = A(t)J_\lambda(t)\Big|_a^b - \int_a^b A'(t)J_\lambda(t)dt$$

Using our results for $A \equiv 1$ for $J_\lambda(t)$, we have

$$\Big|A(t)J_\lambda(t)\Big|_a^b\Big| = \Big|A(b)\int_a^b e^{i\lambda\phi(x)}dx\Big| \leq |A(b)|\frac{8}{\sqrt{\min_{x\in[a,b]}|\phi''(x)|}} \cdot \frac{1}{\sqrt{|\lambda|}}$$

and

$$\int_a^b |A'(t)||J_\lambda(t)|dt \leq \int_a^b |A'(t)|\frac{8}{\sqrt{\min_{x\in[a,b]}|\phi''(x)|}} \cdot \frac{1}{\sqrt{|\lambda|}}dt$$

□

Corollary 1.5. *Assume that the amplitude A and the phase function ϕ are smooth, and that ϕ has finitely many critical points, all of them non-degenerate. Then*

$$I(\lambda) \ll \frac{1}{\sqrt{\lambda}}, \quad \lambda \to +\infty$$

the implied constant depending on A and ϕ.

Proof. To use van der Corput's Lemma, we use a smooth partition of unity to write

$$\mathbf{1}_{[0,2\pi]} = \sum_j \psi_j$$

where ψ_j are smooth, the support of each contains at most one of the critical points, and when the support does contain a critical point x_0 (at which $\phi''(x_0) \neq 0$), we take the support sufficiently small so that $\phi'' \neq 0$ on all of $\operatorname{supp}\psi_j$, while the remaining ψ_j are supported away from the critical points. Hence we can write

$$I(\lambda) = \sum_j I_j(\lambda), \qquad I_j(\lambda) = \int_0^{2\pi} \psi_j(t)A(t)e^{i\lambda\phi(t)}dt.$$

To bound the integrals $I_j(\gamma)$ where the support of ψ_j does not include any critical points, we use the principle of NON-stationary phase, with a smooth amplitude and a phase function so that $|\phi'| \geq c_j > 0$ where $c_j = \min(|\phi'(x)| : x \in \operatorname{supp}\psi_j)$ to bound

$$I_j(\lambda) \ll \frac{1}{|\xi|^N}, \quad \forall N \geq 1.$$

For j such that $\operatorname{supp}\psi_j$ contains a critical point (unique by assumption), we can Corollary 1.4 with $k=2$, since we have taken the support of such ψ_j so that $\phi''\neq 0$ on $\operatorname{supp}\psi_j$. Hence for such j we have

$$I_j(\lambda) \ll \frac{1}{\sqrt{\lambda}}.$$

Altogether we have proven $I(\lambda)\ll \lambda^{-1/2}$. □

1.3. **An asymptotic expansion: The method of stationary phase.** It is possible to go from upper bounds to asymptotic expansions. I will quote a typical result (we will not explicitly use this).

Theorem 1.6. *Assume that the amplitude $A\in C_c^\infty(\mathbb{R})$ is smooth and compactly supported, and that the phase function (real valued and smooth) has a single critical point $x_0\in\operatorname{supp}A$, which is non-degenerate: $\phi'(x_0)=0$, $\phi''(x_0)\neq 0$. Then*

$$I(\lambda)\sim e^{i\frac{\pi}{4}\operatorname{sign}(\phi''(x_0))}A(x_0)\sqrt{\frac{2\pi}{|\phi''(x_0)|}}\cdot\frac{e^{i\lambda\phi(x_0)}}{\sqrt{\lambda}},\quad \text{as } \lambda\to+\infty.$$

1.4. **An application: The Fourier transform of the unit disk.** We want to bound the Fourier transform of the unit disk $B(0,1)\subset\mathbb{R}^2$: If $\chi(x)=1$, $|x|\leq 1$, and is zero otherwise, the Fourier transform is (we have dropped the factor of -2π from the exponent)

$$\widehat{\chi}(\xi)=\int_{|x|\leq 1} e^{i\langle\xi,x\rangle}dx,\qquad \xi\in\mathbb{R}^2.$$

Proposition 1.7.

$$\widehat{\chi}(\xi)\ll\frac{1}{1+|\xi|^{3/2}}.$$

Proof. We convert the 2-dimensional integral to a one-dimensional integral, to which we can apply the van der Corput bound, by using Green's theorem. Recall that Green's theorem says that for a bounded planar domain D, with a piecewise smooth boundary ∂D, we have for A, $B\in C^2(\mathbb{R}^2)$,

$$\int_D\Big(\frac{\partial B}{\partial x}-\frac{\partial A}{\partial y}\Big)dx\wedge dy=\oint_{\partial D}Adx+Bdy$$

where the line integral over the boundary ∂D is taken counterclockwise.

For us, $D=B(0,1)$ is the unit disk, with boundary $\partial D=S^1$ the unit circle, and we want to find A,B so that

$$\frac{\partial B}{\partial x}-\frac{\partial A}{\partial y}=e^{i(ax+by)},\quad \xi=(a,b)\neq 0.$$

A solution is to take

$$A=\frac{ibe^{i(ax+by)}}{|\xi|^2},\quad B=\frac{-iae^{i(ax+by)}}{|\xi|^2}.$$

Hence we find

$$\widehat{\chi}(\xi)=\frac{i}{|\xi|^2}\int_{\partial D}e^{i(ax+by)}\,(bdx-ady).$$

Parameterizing $\partial D = S^1$ by arc-length: $\gamma(t) = (\cos t, \sin t)$, which runs counter-clockwise if t runs from 0 to 2π, we obtain

$$\widehat{\chi}(\xi) = \frac{i}{|\xi|^2} \int_0^{2\pi} e^{i(a\cos t + b\sin t)}(-b\sin t - a\cos t)dt = \frac{i}{|\xi|} I(|\xi|)$$

where

$$I(\lambda) = \int_0^{2\pi} A_\xi(t) e^{i\lambda\phi_\xi(t)} dt$$

is an oscillatory integral, with amplitude

$$A_\xi(t) = \langle \dot\gamma(t), \frac{\xi^\perp}{|\xi|}\rangle = \frac{-a\cos t - b\sin t}{\sqrt{a^2+b^2}}$$

where $\xi^\perp = (b, -a)$, and phase function

$$\phi_\xi(t) = \langle \gamma(t), \frac{\xi}{|\xi|}\rangle = \frac{a\cos t + b\sin t}{\sqrt{a^2+b^2}}.$$

We want to apply our estimates on oscillatory integrals to this case. Note that both the amplitude and the phase function depend on $\xi/|\xi|$, so it is important to make sure that our estimates our uniform in $\xi/|\xi| \in S^1$.

The phase function has two critical points $\phi'(t) = \langle \dot\gamma(t), \xi/|\xi|\rangle = 0$, when $\dot\gamma(t) \perp \xi$, i.e. when $\dot\gamma(t) = \pm\xi^\perp/|\xi|$ (note $\dot\gamma$ is a unit vector), with $\xi^\perp = (b, -a)$, say at t_0 and therefore also at $t_0 + \pi$.

We claim that these critical points are non-degenerate: The second derivative of the phase function is

$$\phi''(t_0) = \langle \ddot\gamma(t_0), \xi\rangle.$$

Now further note that $\langle \ddot\gamma, \dot\gamma\rangle = 0$ (which follows by direct computation or better yet by differentiating $|\dot\gamma|^2 = 1$, which is arc-length parameterization condition), and since also $\xi \perp \dot\gamma(t_0)$, we must have

$$\ddot\gamma(t_0) = \pm\kappa\frac{\xi}{|\xi|}$$

where $\kappa = |\ddot\gamma(t_0)|$; note that here $|\ddot\gamma| \equiv 1$ so $\kappa = 1$ and hence[2]

$$\phi''(t_0) = \langle \ddot\gamma(t_0), \frac{\xi}{|\xi|}\rangle = \pm\langle \frac{\xi}{|\xi|}, \frac{\xi}{|\xi|}\rangle = \pm 1 \neq 0.$$

To use van der Corput's Lemma, we use a smooth partition of unity to write

$$\mathbf{1}_{[0,2\pi]} = \psi_1 + \psi_2 + \psi_3$$

where ψ_j are smooth, ψ_1 is supported in say $(t_0 - 0.1, t_0 + 0.1)$, ψ_1 is supported in $(t_0 + \pi - 0.1, t_0 + \pi + 0.1)$ and ψ_3 is supported away from the critical points $t_0, t_0 + \pi$. This gives

$$I(\lambda) = I_1 + I_2 + I_3$$

with

$$I_j(\lambda) = \int_0^{2\pi} \psi_j(t) A(t) e^{i\lambda\phi(t)} dt.$$

[2]For general D, $|\ddot\gamma| = \kappa$ will be the curvature of ∂D; if we assume that the boundary ∂D has nowhere vanishing curvature, then this computation shows that the critical points are all non-degenerate.

For the integrals I_1, I_2 we can use van der Corput's Lemma to deduce that

$$|I_1(|\xi|)| \leq \frac{8}{(\min_{x\in[t_0-0.1,t_0+0.1]} |\phi^{''}(x)|)^{1/2}} \cdot \Big(|A(t_0+0.1)| + \int_0^{2\pi} |A'(t)|dt\Big)\frac{1}{|\xi|^{1/2}}.$$

We have

$$|A(t)| = |\langle \dot\gamma(t), \frac{\xi}{|\xi|}\rangle| \leq |\dot\gamma| \cdot |\frac{\xi}{|\xi|} = 1$$

and likewise $|A'(t)| \leq 1$. Since $\phi''(t_0) = \pm 1$, we have

$$\min(|\phi''(t)| : t_0 - 0.1 < t < t_0 + 0.1) > c_0$$

for some positive constant c_0. Hence we deduce from van der Corput's Lemma that there is some $C > 0$, independent of ξ, so that

$$|I_1(|\xi|)|, |I_2(|\xi|)| \leq \frac{C}{|\xi|^{1/2}}.$$

To bound the third integral $I_3(\gamma)$, we use the principle of NON-stationary phase, with a smooth amplitude and a phase function so that $|\phi'| \geq c_3 > 0$ (uniformly in ξ) to bound

$$I_3 \ll \frac{1}{|\xi|^N}, \quad \forall N \geq 1.$$

Altogether we obtain that uniformly in $|\xi| \geq 1$,

$$I(\lambda) \leq \frac{c_4}{\sqrt{\lambda}}$$

and hence

$$|\widehat{\chi}(\xi)| \leq \frac{c_4}{|\xi|^{3/2}}.$$

□

Exercise 2. *Show that in dimension 3, we have an estimate for the Fourier transform of the unit ball*

$$\int_{|x|\leq 1} e^{i\langle x,\xi\rangle} d^3x \ll \frac{1}{1+|\xi|^2}, \quad \xi \in \mathbb{R}^3.$$

Hint: Here we can directly evaluate the Fourier transform in elementary terms!